# Intermittency in turbulence: Computing the scaling exponents in shell models

Roberto Benzi,[1,2] Luca Biferale,[1,2] Mauro Sbragaglia,[1,2] and Federico Toschi[2,3]
[1]*Dipartimento di Fisica, Università "Tor Vergata," Via della Ricerca Scientifica 1, I-00133 Roma, Italy*
[2]*INFM, Sezione di Roma "Tor Vergata," Via della Ricerca Scientifica 1, I-00133 Roma, Italy*
[3]*Istituto per le Applicazioni del Calcolo, CNR, Viale del Policlinico 137, I-00161 Roma, Italy*
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We discuss a stochastic closure for the equation of motion satisfied by multiscale correlation functions in the framework of shell models of turbulence. We present a plausible closure scheme to calculate the anomalous scaling exponents of structure functions by using the exact constraints imposed by the equation of motion. We present an explicit calculation for fifth-order scaling exponent by varying the free parameter entering in the nonlinear term of the model. The same method applied to the case of shell models for Kraichnan passive scalar provides a connection between the concept of zero-modes and time-dependent cascade processes.

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## I. INTRODUCTION

Since the fundamental work by Kolmogorov, it has been recognized that a consistent theory for the statistical properties of turbulence should quantitatively explain intermittency. In the last ten years many important steps have been taken to provide a consistent picture of intermittency in turbulence. First, experimental measurements and a new systematic way to analyze data have shown the universal feature of intermittency [1,2]. Second, a well-defined theory has been proposed to compute anomalous scaling for a class of *linear* problems, i.e., the case of Kraichnan passive scalar [3]. In the latter case, the notion of *zero modes* provided a theoretical framework for many fundamental properties of intermittency. Yet, we are still looking for defining a suitable strategy for a quantitative computation of intermittency in the full nonlinear problem, namely, the Navier-Stokes equations. The problem of anomalous scaling must be divided into two steps. First, we need to clean it from all unwanted difficulties, trying to focus on the main physical mechanism leading to small-scale intermittency and to its connections with the nonlinear structure of the equation of motion. This is the main goal of this paper. We show that the anomalous scaling of small-scale velocity fluctuations of a shell model of turbulence can be derived from the equation of motion. The result is based on a stochastic closure. A second, more ambitious goal, is to extend this result to the full complexity of Navier-Stokes equations. Some comments on the latter problem are also proposed in the conclusions.

Let us make a few general comments on the nature of the problem we are facing. We are interested in the (universal) features of the statistical properties of the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ in a homogeneous and isotropic turbulent flow. Experimental data and theoretical ideas suggest that these universal properties are related to velocity fluctuations at scales much smaller than the energy input scale $L$. More precisely, we want to compute the simultaneous multipoint correlation functions $C^{(n)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n)=\langle v_{i_1}(\boldsymbol{x}_1,t)v_{i_2}(\boldsymbol{x}_2,t),\ldots,\ldots,v_{i_n}(\boldsymbol{x}_n,t)\rangle$ for scale separations $|\boldsymbol{x}_i-\boldsymbol{x}_j|$ much smaller than $L$. Our task must be performed by using the Navier-Stokes equations. Equations of motion provide a relation among the (infinite) sets of simultaneous correlation functions $C^{(n)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n)$:

$$0=\frac{d}{dt}C^{(n)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n)=\Gamma[C^{(n+1)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n)], \tag{1}$$

where we have assumed stationarity and with $\Gamma$ we denote the integrodifferential linear operator derivable explicitly from the Navier-Stokes equations.

For Kraichnan model the equivalent of the above hierarchy is closed, order by order in the correlation functions, allowing for a perturbative calculation of some statistical properties. In the full Navier-Stokes problem one can show that Eqs. (1) do not form a closed set of equations, rather it should be considered as a constraint for the complete solution. Actually, the fundamental quantities for studying intermittency in turbulence involve also temporal information from multitime correlation functions $C^{(n)}(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2;\ldots;\boldsymbol{x}_n,t_n)=\langle v_{i_1}(\boldsymbol{x}_1,t_1)v_{i_2}(\boldsymbol{x}_2,t_2)\cdots v_{i_n}(\boldsymbol{x}_n,t_n)\rangle$. Namely, we need to look for the solution of the problem

$$\frac{\partial^k}{\partial t_1\cdots\partial t_k}C^{(n)}(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2;\ldots;\boldsymbol{x}_n,t_n)=\Sigma[C^{(n+1)}], \tag{2}$$

with $k\leqslant n$ and $\Sigma$ is a functional of the time-dependent correlation functions of order $n+1$ depending on $n+1$ velocity fields at $n$ different times. The fundamental question we are facing is which are, if any, the physical informations we should use in order to *solve* Eq. (2). In some broad sense, not being able to solve Eq. (2) by any kind of "brute force" attempt, we still need to understand which are the correct "order parameters" we should consider to find out a systematic way to compute a solution of Eq. (2).

We argue that a strategy to compute the solutions of the multitime hierarchy (2) may be outlined by first finding a "physically consistent" solution for the simultaneous hierarchy (1) which can be used as the starting point for successive approximations. By physically consistent we mean that the solution should respect the phenomenological constraints imposed by the Navier-Stokes equations and in particular by its time-space scaling properties. This is the main idea pursued in this paper. More precisely, we will discuss how far we can provide a quantitative computation of intermittency based on the following three main points: (i) we only use the constraints coming from the simultaneous Eqs. (1); (ii) we look for the solutions of Eq. (1) by assuming that the out-of-equilibrium statistical properties of the velocity field can be obtained by a suitable time-dependent stochastic process; (iii) we shall restrict ourselves to nonlinear shell models [1,4].

Having discussed in details the motivation of point (i), let us briefly comment on point (ii). Random multiplicative processes have been often used in literature as a simple mathematical tool to describe anomalous scaling properties of turbulent flows [5]. Only a few attempts have successfully linked cascade-multiplicative process with the structure of the equation of motion [6]. Recently, the concept of random multiplicative process has been enlarged by including nontrivial time dynamics [7,8]. In particular, the choice of time dynamics can be done in order to satisfy the Navier-Stokes temporal scaling (in a Lagrangian reference frame). Moreover, it has been shown that time dynamics affects, in a non trivial way, also the spatial scaling of $C^{(n)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n)$. Our strategy is to employ the statistical constraint of time-dependent random multiplicative process to look for a solution of Eqs. (1). The theory of time-dependent random multiplicative process is in its infancy. Only few exact results have been obtained so far. One can wonder why we need a time-dependent stochastic process as a tool to describe equal time correlation functions. The answer is that the shape of the correlation functions is strongly dependent on the time dynamics [9]. The hope is that, by using the dynamical scaling required by the Navier-Stokes equations, we can already obtain a good approximation to the real solutions. Finally, we want to comment on point (iii). Shell models provide the simplest model to check our strategy and to compare our physical ideas with clean numerical simulations in the asymptotic regime of large Reynolds numbers.

Even with the approximations defined in points (i)–(iii), the problem of computing the universal anomalous scaling in turbulence is equivalent to solving a functional equation, i.e., each Eq. (1) defines, for any order, a constraint for the probability distribution. We will limit ourselves to the lowest, nontrivial, order such as to be able to push the calculation analytically as much as possible.

The paper is organized in the following way. In Sec. II we briefly recall the basic properties of time-dependent random multiplicative process. In Sec. III we address the problem of anomalous scaling in Kraichnan shell models of passive scalars. There, we present a rederivation in the framework of stochastic closure of an exact result for the anomalous scaling of fourth-order structure function. Thus, we are able to connect the mathematical notion of *zero modes* with the *cascade mechanism* described by the time-dependent multiplicative process. In Sec. IV we extend the stochastic closure used for the passive scalar case to the fully nonlinear model. We discuss at length both similarities and differences between the two cases and we present, to our knowledge, the first attempt to calculate the fifth-order scaling exponents by varying the values of the free parameter in the shell model. Conclusions will follow in Sec. V.

## II. TIME-DEPENDENT RANDOM MULTIPLICATIVE PROCESS

Let us first review the main ingredients of time-dependent multiplicative process (TRMP) [8]. We introduce a set of reference scales, $\ell_n=\ell_0 2^{-n}$, and a set of velocity increments at scales, $\ell_n$: $\delta_n v \sim v(x+\ell_n)-v(x)$. The basic idea of random *time-independent* multiplicative process is to assume that statistical properties of $\delta_n v$ can be obtained by

$$\delta_{n+1} v = A_{n+1}\delta_n v, \tag{3}$$

where $A_i$ are i.i.d. (independent identically distributed) random variables with a time-independent—bare— probability $P(A)$. Time-independent multiplicative process as Eq. (3) has been widely used in the past to mimics the spatial distribution of velocity fluctuations in turbulence and the multifractal energy dissipation measure [1]. Recently, also an attempt to match the stochastic multiplicative model with the deterministic structure of the equation of motion of a shell model of turbulence has been presented [6]. Despite the success in reproducing the cascade phenomenology, time-independent multiplicative process cannot capture the subtle complexity of the spatial and temporal behavior of Navier-Stokes equations (in a Lagrangian reference frame). For example, multiscale correlation functions of the kind $\langle \delta_n v\, \delta_{n'} v\rangle$ are well reproduced only asymptotically for large-scale separations, $n' \gg n$ [10]. The problem is that simple time-independent random multiplicative processes do not take into account the time dynamics, i.e., they are not constrained by the equation of motion. To overcome this difficulty, a new class of time-dependent stochastic multiplicative process (TRMP) have been proposed [7,8]. Basically, the idea is to mimic the temporal constraints imposed by the structure of the Navier-Stokes equations, $\partial_t v \sim v\,\partial v$, by requiring that the multiplicative structure (3) is satisfied for the random time interval $\tau_{n+1}=\ell_n/(\delta_n v)$. To build the temporal dynamics we proceed as follows. We extract the instantaneous multiplier $A_n$, connecting the amplitudes of two velocity fluctuations at adjacent scales, $\delta_n v = A_n \delta_{n-1} v$, with a given probability $P(A)$, independent from the scale, $\ell_n$, and we keep it constant for a time interval $[t,t+\tau_n]$, with $\tau_n=\ell_n/(\delta_n v)$ being the local instantaneous eddy-turn-over time. Thus, for each scale $\ell_n$, we introduce a time-dependent random process $A_n(t)$ which is piecewise constant for a random time interval $[t_n^{(k)},t_n^{(k)}+\tau_n]$, if $t_n^{(k)}$ is the time of the $k$th jump at scale $n$. The corresponding velocity field at scale $n$, in the time interval $t_n^{(k)}<t<t_n^{(k)}+\tau_n$, is given by the simple multiplicative rule:

$$\delta_n v(t) = A_n(t)\,\delta_{n-1} v(t_n^{(k)}). \tag{4}$$

What is important to notice is that at each jumping time $t_n^{(1)},t_n^{(2)},\ldots,t_n^{(k)},\ldots$ only the velocity field at the corresponding shell $n$ is updated, i.e., information across different scales propagates with a finite speed. In this way we reproduce the phenomenology of the nonlinear evolution of Navier-Stokes dynamics: $\partial_t v \propto v\,\partial v$. Multipliers at different scales develop correlations through the time dependency. From now on, we will denote with $\overline{\cdots}$ averages with respect to the stochastic process and with $\langle \cdots \rangle$ averages over the chaotic deterministic dynamics of the shell model.

A more detailed numerical and theoretical analysis of the statistical properties of TRMP can be found in Ref. [8]. The possibility to reproduce the *single-time* statistical properties of TRMP by a Gibbs-like measure has also been recently discussed in Ref. [9].

## III. TRMP AND THE KRAICHNAN MODEL

We start our work defining the relationship between the Kraichnan shell model for passive scalar and time-dependent random multiplicative process. We first review the model and see how the computation of the anomalous exponents can be rigorously done in this case. The model is defined as follows [11,12]. Passive increments are described on a discrete subset of wave numbers (shells) $k_n = k_0 \lambda^n$, by a complex variable $\theta_n(t)$, which satisfy the equations $(n=1,2,\ldots,N)$

$$\left[\frac{d}{dt} + \kappa k_n^2\right]\theta_n(t) = i[c_n \theta^*_{n+1}(t) u^*_n(t) + b_n \theta^*_{n-1}(t) u^*_{n-1}(t)] + \delta_{1n} f(t), \tag{5}$$

where the star denotes complex conjugation and $b_n = -k_n, c_n = k_{n+1}$ are chosen such as to impose energy conservation in the zero diffusivity limit. The intershell ratio must be taken such as $\lambda > 1$. Boundary conditions are defined as $u_0 = \theta_0 = 0$. The forcing term $\delta_{1n} f(t)$ is Gaussian and delta correlated: $\langle f(t) f(t')\rangle = F_1 \delta(t-t')$. It acts only on the first shell. Kraichnan models of passive advection assume that each velocity variable $u_n(t)$ is a complex Gaussian and white-in-time stochastic process, with a variance which scales as $\langle u_m(t) u^*_n(t')\rangle = \delta(t-t')\delta_{nm} d_m$, $d_m = k_m^{-\xi}$. The cross correlation between the advecting velocity variables and the passive variable can be rewritten in terms of passive correlations only, when the velocity field is a white-in-time Gaussian variable. Thus, all equations for all passive structure functions are closed [14,11,12]. The goal is to calculate the scaling exponents $H(p)$ of the $p$th order passive structure functions as defined by

$$\langle |\theta_n|^p \rangle \sim k_n^{-H(p)}.$$

We concentrate on the nonperturbative analytic calculation of the fourth-order structure function $P_{nn} = \langle (\theta_n \theta^*_n)^2 \rangle \propto k_m^{-\zeta_4}$ (the lowest order with nontrivial anomalous scaling). The closed equation satisfied by $P_{nq} = \langle (\theta_n \theta^*_n)(\theta_q \theta^*_q)\rangle$ is

$$\dot{P}_{nq} = (\delta_{1,n} E_n + \delta_{1,q} E_q) F_1 - \kappa (k_n^2 + k_q^2) P_{nq} + \{ -P_{nq} c_n^2 d_n [(1 + \delta_{q,n+1}) + \lambda^{\xi-2}(1+\delta_{q,n-1})] + P_{n+1,q} c_n^2 d_n (1+\delta_{q,n}) + P_{n-1,q} b_n^2 d_{n-1}(1+\delta_{q,n}) + (q \leftrightarrow n)\}, \tag{6}$$

where $E_n = \langle \theta_n \theta^*_n \rangle = E_0 k_n^{\xi-2}$, i.e., the second-order scaling exponent is given by $H(2) = 2-\xi$. The above equation can be elegantly rewritten in the operatorial form:

$$\dot{P}_{nq} = \mathcal{I}_{nq,n'q'} P_{n'q'} + \kappa \mathcal{D}_{nq,n'q'} P_{n'q'} + \mathcal{F}_{nq}, \tag{7}$$

where we have explicitly separated the inertial $\mathcal{I}$ from the dissipative $\mathcal{D}$ part of the linear operator and where the nonhomogeneous term composed by the forcing and by the second-order passive structure functions is summarized in the expression $\mathcal{F}_{nq}$.

It is useful to highlight in the two-scale correlation function $P_{n,n+l}$ the dependency from the scale separations by introducing the set of variables $C_l$:

$$P_{n,n+l} = C_l P_{n,n}. \tag{8}$$

From basic scaling principle one may argue that the asymptotic scaling behavior is given by the so-called fusion rules [15,16,7,10]:

$$P_{n,n+l} \sim \frac{\langle |\theta_{n+l}|^2\rangle}{\langle |\theta_n|^2\rangle} \langle |\theta_n|^4\rangle, \quad l \to \infty, \tag{9}$$

which means that $C_l \sim C_\infty k_l^{\xi-2}$ for $l$ positive and large enough. Similarly, for $l$ negative, we may write

$$P_{n,n-l} = D_l P_{n,n}, \tag{10}$$

where now the asymptotic behavior of $D_l$ feels the fourth-order scaling behavior $D_l \sim D_\infty k_l^{-(\xi-2)-\rho_4}$ for $l$ large enough, with $\rho_4 = H(4) - 2H(2)$ being the anomaly of the fourth-order scaling exponent. The two sets of variables $D_l$ and $C_l$ are not independent. By introducing the notation, $x = \lambda^{\xi-2}$ and $R = \lambda^{\rho_4}$, one may rewrite both of them as a function of a new set of variables $\Gamma_l$ defined as $C_l = \Gamma_l x^l$ and $D_l = \Gamma_l/(xR)^l$ [12]. The assumption that fusion rules are satisfied is the only crucial point in computing the zero modes. The existence of fusion rules implies that correlation functions show scaling in the inertial range.

The infinite set of equations for the inertial-range *zero-mode* of Eq. (6), $\mathcal{I}_{mq,m'q'} P_{m'q'} = 0$, can be rewritten in the following form:

$$A_0(x,R) + B_{0,1}(x,R)\Gamma_1 = 0, \quad q = n, \tag{11}$$

$$A_1(x,R) + B_{1,1}(x,R)\Gamma_1 + B_{1,2}(x,R)\Gamma_2 = 0, \quad n = q+1, \tag{12}$$

$$B_{n,n-1}(x,R)\Gamma_{n-1} + B_{n,n}(x,R)\Gamma_n + B_{n,n+1}(x,R)\Gamma_{n+1} = 0, \quad n > q+1, \tag{13}$$

where the functions $A_0, A_1, B_{i,j}$ are known functions of $x$ and $R = \lambda^{\rho_4}$. The computation of the zero modes means to find out the numbers $R$ and $\Gamma_i$ which solves Eqs. (11), (12), and (13). Let us remark that for any given total number of shells, $N$, we have $N+1$ equations and $N+2$ unknowns which are given by the $\Gamma_i$ for $i = 1, \ldots, N+1$ plus the parameter directly affected by the fourth-order anomalous exponent

$R(\rho_4)$. Thus, it is impossible to find a solution unless some extra information is added to our problem. This information can be found by observing that for large $n$ the functions $B$ appearing in Eq. (13) become constants independent of both $x$ and $R$. In the limit of large $n$, defining

$$Z_n = \frac{\Gamma_{n+1}}{\Gamma_n},$$

one finds that Eq. (13) can be rewritten as

$$Z_{n+1} = \Phi(Z_n), \tag{14}$$

where the explicit form of $\Phi$ is given in Ref. [12]. Map (14) possesses a fixed point $Z^* = 1$ for large shell index $n$ which corresponds to the fact that $\Gamma_n$ reaches a plateau for large $n$, i.e., to the fact that fusion rules are asymptotically satisfied. The crucial point is to observe that $Z^*$ is a stable fixed point for the inverse of $\Phi$, i.e., for

$$Z_l = \Phi^{-1}(Z_{l+1}). \tag{15}$$

The stability of $Z^*$ for Eq. (15) allows us to compute the values of $Z_n$ for small $n$, i.e., we start with $Z_\infty = 1$ and then we compute $Z_m$ by using Eq. (13) up to $m = 2$. In this way we can compute $Z_2$ as a function of $R$ and $x$. Thus Eqs. (11) and (12) become

$$A_0(x,R) + B_{0,1}(x,R)\Gamma_1 = 0, \tag{16}$$

$$A_1(x,R) + B_{1,1}(x,R)\Gamma_1 + B_{1,2}(x,R) Z_2(x,R)\Gamma_1 = 0. \tag{17}$$

Equations (16) and (17) have two unknowns, namely, $\Gamma_1$ and $R$, for two equations and, therefore, one can find a solution. The analytical solution turns out to be in perfect agreement with the numerics both for the fourth-order object described here [12] and for higher-order correlations [13]. This ends the review of the analytical results previously obtained on the model.

The solution of the Kraichnan shell model for passive scalar provides us the rigorous computation of the zero modes. We want now to understand whether the computation of the zero modes can be pursued by using the concept of time-dependent random multiplicative processes. In order to define a suitable TRMP to define the case of the Kraichnan passive scalar we take the usual TRMP discussed in the preceding section for the updating of scalar fluctuations at two adjacent scales:

$$\theta_{n+1}(t) = A_{n+1}(t)\,\theta_n(t), \tag{18}$$

where $A_i$ are i.i.d. random variables with a time-independent probability $P(A)$. The only difference with the TRMP for the velocity field is that now the updating time of the multipliers must satisfy the dynamical law: $\partial_t \theta \sim v \partial \theta$. Thus, we need to update the multiplicative structure (18) at the random time interval $\tau_{n+1} = 1/(k_n u_n)$, uncorrelated from the probability distribution of the multipliers themseleves (scalars are passive). Moreover, because the advecting field is a Gaussian field with correlation functions proportional to $k_n^{-\xi}$ one can deduce that $\tau_n$ is not a random time and should be chosen as $\tau_n = k_n^{\xi-2}$. One can show that such a class of TRMP predicts a nontrivial behavior of the fusion rules coefficients $C_l, D_l$. Moreover, the detailed behavior of the fusion rules coefficients are determined by the spatial intermittency, i.e., the constraint $\partial_t \theta \sim v \partial \theta$ between temporal and spatial scalings induced by the Navier-Stokes structure in a Lagrangian reference frame is satisfied. This is the crucial point we need to use to solve our problem. We can summarize our discussion in the following way. TRMP provides us with a relationship between each fusion rule coefficient $C_l$, $D_l$ and the anomalous exponents. In this way we are building a stochastic closure for Eqs. (11), (12), and (13). Moreover, the time and spatial dependencies of the stochastic process are consistent with the structure of the deterministic equation of motion. We want to show here that besides the exact method discussed before the stochastic closure through the TRMP also works.

In the following we assume that the—bare—probability $P(A)$ is log normal. We are aware that log-normal probability distributions are not consistent with the anomalous scaling of turbulent flows or shell models for large orders, even for the case of the Kraichnan shell model. However as far as we are interested, to compute $H(p)$ for rather small $p$, log normality is a reasonable approximation which simplifies the analytical computations. Because we know that $H(2) = 2 - \xi$, the probability distribution $P(A)$ depends only on a single unknown parameter $\sigma$ which describes the variance of the log-normal fluctuations. By using the exact solution previously discussed, for each value of $\xi$ we can compute the value of $H(4)$. Thus for each value of $\xi$ we can fix the parameters of the log-normal distribution in order to reproduce the anomalous exponent. We can next simulate the TRMP numerically and compute the value of the fusion rules coefficients $\Gamma_l$. The most sensitive test is made by comparing the prediction on the asymptotic values of $\Gamma_l$ for large $l$ which we denote by $\Gamma_\infty$ (notice that $\Gamma_0 = 1$ by definition) as extracted from the computation of the zero mode and from the TRMP.

Before doing a direct comparison between the TRMP and the exact solution, we need to discuss another subtle point. The definition of a random multiplicative process shows an extra degree of freedom that is fixed neither by the scaling properties nor by the dynamical scaling. To be more precise, in the case of the Kraichnan model, it is possible to define $\theta$ as

$$\theta_n(t) = g_n \prod_i^n A_i(t),$$

where $g_n$ are i.i.d. random variable for any scale $n$. Because the probability distribution of $g_n$ does not depend on $n$, then the scaling properties of $\theta_n$ does not depend on $g_n$. However, the fusion rules coefficients $\Gamma_l$ do depend on $g_n$. In particular the quantity $\Gamma_\infty$ depends on $g_n$ as

$$\Gamma_\infty(g=1) \rightarrow \Gamma_\infty(g)\frac{\langle g^2\rangle^2}{\langle g^4\rangle}.$$

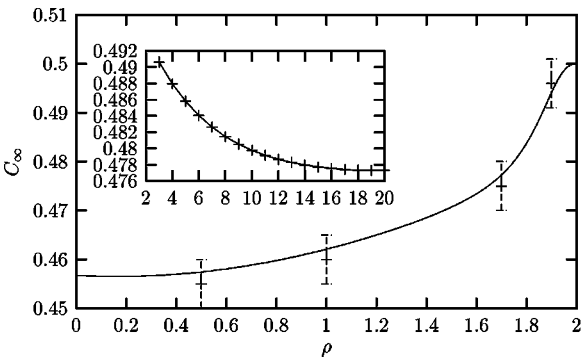


FIG. 1. Results for the asymptotic value $C_\infty$ of the fusion rules coefficients for the Kraichnan model. For different values of $\xi$, the result obtained by the analytic computation of the zero modes (continuous line) is compared with the estimate (+) obtained using the TRMP. Inset: plot of $\Gamma_n$ vs $n$ in the case of the TRMP. The value of $\Gamma_n$ is multiplied by the factor $\frac{1}{2}$.

Thus it seems that in our way to apply time-dependent multiplicative process to the Kraichnan shell model we are not able to fix the fusion rules coefficients. This is rather disappointing because we start all our analysis by pointing out that the shape of the fusion is determined by the time dynamics in a TRMP. However, the function $g$ and its probability distribution should not depend on intermittency itself. In particular, it is relatively easy to compute $g$ and its probability distribution for the Kraichnan model by observing that for $\xi=2$ all scaling exponents $H(p)=0$ and $\rho_p=0$, as already observed in the work by Kraichnan [14]. Using this information in Eqs. (11), (12), (13) we find that $\Gamma_1=1/2$. Similarly, we can generalize this information for all fusion rules coefficients. This constraint can be satisfied only by a suitable choice of $g_n$. It turns out that in the Kraichnan shell model it is equivalent to choose $g_n$ to be Gaussian. Thus the value of $\Gamma_\infty$ should be multiplied by 2 in order to compare it with the TRMP. The comparison is shown in Fig. 1.

As one can see the results are in extremely good agreement with the exact solution. The above results provide us with a complete and clear physical intuition of what a *zero mode* is. We have shown that the interpretation of anomalous scaling in terms of *zero modes* is fully compatible with the statistical properties of multiplicative stochastic models. The only missing brick was the importance of temporal dynamics. Anomalous scaling as described by the zero modes of the inertial operator for the *simultaneous* $p$th order structure functions is the outcome of the *time-dependent* energy transfer from large scales to small scales. Here we have shown that a suitable closure based on TRMP is indeed sufficient to calculate the zero mode for fourth-order structure function $\langle|\theta_m|^2|\theta_q|^2\rangle$ in the inertial range.

## IV. NONLINEAR SHELL MODEL

Here we want to understand if a result similar to the one shown in the preceding section still holds true for the nonlinear shell models. This is important because (i) we exploit the possibility to use TRMP in the full nonlinear case; (ii) we can generalize the concept of zero modes; (iii) we find out a way to compute the scaling exponents. The model we used is an improved version of the GOY (Gledzer-Ohkitani-Yamada) model [17,18], proposed in Ref. [19] (see also Ref. [4] for a recent review):

$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n=i[k_nu_{n+1}^*u_{n+2}+bk_{n-1}u_{n+1}u_{n-1}^* +(1+b)k_{n-2}u_{n-2}u_{n-1}]+f_n, \tag{19}$$

where $u_n$ is a complex variable representing the velocity fluctuations at wave number $k_n$, where $k_n=2^nk_0$. Numerical simulations show that the variables $u_n$ exhibit anomalous scaling for $-1\leqslant b\leqslant 0$ , namely,

$$S_p(n)\equiv\langle|u_n^p|\rangle\sim k_n^{-\zeta(p)}, \tag{20}$$

where $\zeta(p)$ is a nonlinear function of $p$. Numerically it is observed that the anomalous scaling behavior depends on the parameter $b$ and it does not depend on the specific form chosen for the large-scale forcing $f_n$.

### Computing the scaling exponents

By defining $Q_n(t)=u_n(t)u_n^*(t)$, we start by searching a solution of the equation obeyed by the simultaneous fourth-order correlation in the limit of zero viscosity , i.e., neglecting dissipative effects:

$$\frac{d}{dt}\langle Q_mQ_n\rangle=k_{n+1}\langle Q_mW_{n+1}\rangle+bk_n\langle Q_mW_n\rangle -(1+b)k_{n-1}\langle Q_mW_{n-1}\rangle+(n\leftrightarrow m)=0, \tag{21}$$

where we have introduced the flux variable given by the third-order object $W_n=\mathrm{Im}(u_{n+1}^*u_nu_{n-1})$. Equations (21) can be written as an infinite set of linear equations for the fifth-order correlation function: $\langle Q_nW_m\rangle$. First, we can pick out the asymptotic behavior given by the usual fusion rule:

$$\langle Q_{n+l}W_n\rangle=D_lk_l^{-\zeta(2)}S_5(n),\quad \langle W_{n+l}Q_n\rangle=C_lk_l^{-\zeta(3)}S_5(n). \tag{22}$$

Fusion rules are a general property of the correlation functions in turbulence, predicted by random multiplicative processes and verified with very good accuracy in laboratory experiments [16,10]. In particular, it is known that for large $l$, $C_l$ and $D_l$ are no longer dependent on $l$.

How to obtain information on the behavior of $D_l$ and $C_l$? By restricting ourselves to equal time correlation functions [i.e., Eqs. (1)] there is no hope to close the problem we are facing and it is impossible to get any useful information by using Eqs. (21). In order to make progress, we now assume as in Sec. III that the statistical properties of $u_n$ can be described in terms of a time-dependent random multiplicative process. We will now employ the following approximations: (i) we use $C_l$ and $D_l$ only for small $l$, i.e., $l\leqslant 2$; (ii) for small $l$ we can assume that $C_l=D_l$. Using these approximations we can rewrite from Eq. (21) the equations regarding $C_1$ and $C_2$ as follows:

$$\langle Q_n W_n\rangle[C_1+b-(1+b)C_1R^{-1}]=0, \tag{23}$$

$$\langle Q_n W_n\rangle\{[1-(1+b)xR^{-1}]C_2+b(1+x)C_1 + [Rx-(1+b)]\}=0, \tag{24}$$

where we have introduced the shorthand notation $x=\lambda^{-\zeta(2)}$ for the dependency on the second-order exponents and $R=\lambda^{\zeta(3)+\zeta(2)-\zeta(5)}$ for the dependency on the fifth-order anomaly $\rho_5=\zeta(3)+\zeta(2)-\zeta(5)$. Also here, as in the passive scalar, we have more unknowns than equations. Precisely, once given the second-order exponent $\zeta(2)$, we have two equations and three unknowns, the two fusion rules coefficients for close-by shells, $C_1,C_2$, and the fifth-order anomaly, $R(\rho_5)$. Unfortunately one cannot follow the same—winning—strategy adopted for the passive scalar, because here the equivalent of map (14) is not stable for back iteration. In order to close the problem, we must provide information on the value $Z_2=C_2/C_1$ as a function of $\rho_5$ and $\zeta(2)$. Here is where we want to exploit the TRMP.

In order to apply our strategy, we first need to face the following problem. The structure of the Eqs. (23) we want to close deals explicitly with complex shell variables. Therefore one should define two correlated random processes; one for the amplitude $|u_n|$ and another for the phase of the velocity shell variable. Such a strategy, although feasible, introduces new unknowns which need to be fixed either by using the equations of motion or by using additional information. To not increase the complexity of the problem we look for a simpler and suitable approximation. The key point is that we need to use TRMP just to obtain the quantity $Z_2=C_2/C_1$, i.e., we need to control the ratio

$$\mathcal{R}=\frac{\langle W_{n+2}Q_n\rangle}{\langle W_{n+1}Q_n\rangle}. \tag{25}$$

Defining $u_n=|u_n|\exp(i\phi)$, we can write

$$\mathcal{R}=\frac{\langle|u_{n+3}||u_{n+2}||u_{n+1}||u_n|^2\sin\Delta_{n+2}\rangle}{\langle|u_{n+2}||u_{n+1}||u_n|^3\sin\Delta_{n+1}\rangle}, \tag{26}$$

where $\Delta_n=\phi_n+\phi_{n-1}-\phi_{n+1}$. Expression (26) tells us that, if the correlation between phases and amplitudes is negligible, we can rewrite Eq. (26) as follows:

$$\mathcal{R}=\frac{\langle|u_{n+3}||u_{n+2}||u_{n+1}||u_n|^2\sin\Delta_{n+2}\rangle}{\langle|u_{n+2}||u_{n+1}||u_n|^3\sin\Delta_{n+1}\rangle} \sim\frac{\langle|u_{n+1}|^3|u_n|^2\sin\Delta_{n+2}\rangle}{\langle|u_n|^5\sin\Delta_{n+1}\rangle}, \tag{27}$$

where we have fused the shell variables at scales $n+3$ and $n+2$ with shell variable at scale $n+1$ in the numerator and shell variables $n+2$ and $n+1$ with shell at scale $n$ in the denominator. The above considerations can be formally stated by writing

$$\frac{\langle W_{n+2}Q_n\rangle}{\langle W_{n+1}Q_n\rangle}=\mathcal{K}(k_n,b)\frac{\langle|u_{n+1}|^3|u_n|^2\rangle}{\langle|u_n|^5\rangle}, \tag{28}$$

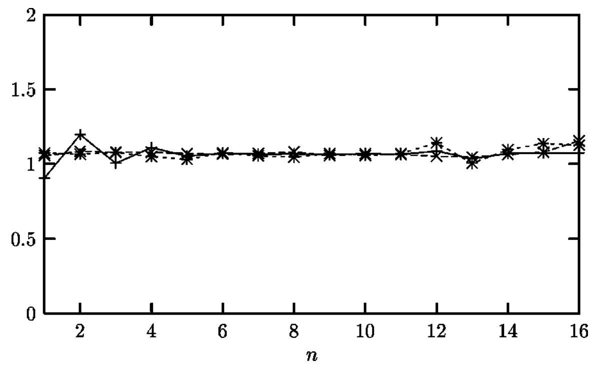


FIG. 2. Plot of $\mathcal{K}(k_n,b)$ vs $n$ computed from simulations of the shell model for the following values of the parameter $b$: $b=-0.4$ (+), $b=-0.6$ (×), $b=-0.7$ (⋆). Only data in the inertial range are shown.

where $\mathcal{K}(k_n,b)$ takes into account the correlation, if any, between the phases $[\sin(\Delta_n)]$ and the amplitude of the shell variables. We expect that the quantity $\mathcal{K}(k_n,b)$, defined in Eq. (28), does not depend on the scale (at least in the inertial range) and might depend on the degree of intermittency, i.e., on $b$. In particular, if $\mathcal{R}$ strongly depends on the correlation between phases and amplitudes of the shell variables, then $\mathcal{K}$ should be strongly dependent on the free parameter $b$ entering in the definition of the nonlinear terms.

The above discussion can be summarized by saying that the quantity $\mathcal{K}(k_n,b)$ is a direct measure of the importance of the cross correlations between shell amplitudes and phases. In order to work out a suitable strategy to apply TRMP as a statistical closure for the nonlinear shell model, we only need that $\mathcal{K}$ is independent of $b$. Let us remark that such a requirement is not equivalent to a "random phase approximation" (which would imply $\mathcal{K}=1$). In the following we shall assume that $\mathcal{K}$ is independent of intermittency corrections, i.e., of $b$. Our assumption is justified by the numerical results shown in Fig. 2. As one can see, the parameter $\mathcal{K}$ is indeed constant, independent of both the shell index and the intermittency intensity as measured by the variation of the parameter $b$ in the equation of motion.

Consequently, we may safely proceed with a simple TRMP based on amplitudes only, using Eq. (28) with $\mathcal{K}=$const still to be determined. Concerning the multipliers distributions (18), as in the case of the Kraichnan shell model, we assume that $P(A)$ is log normal. Let us recall that the exponents $\zeta^{(s)}(p)$ measured from the scaling of the stochastic signal $\overline{|u_n|^p}\sim k_n^{\zeta^s(p)}$ do not coincide with the *bare* scaling exponents as estimated by the instantaneous multiplicative process, $\zeta^{(b)}(p)=-\log_\lambda\langle A^p\rangle$, due to the correlation between the local eddy-turn-over time and the velocity fluctuations, the time dynamics *renormalize* the spatial scaling [8]. This is an extra complication with respect to the passive case. Hereafter we always refer to the bare exponents as $\zeta^{(b)}(p)$ and to those actually measured on the stochastic signal as $\zeta^{(s)}(p)$. We proceed by performing the numerical estimate of the scaling properties of the stochastic signal (4) by changing the parameters of the *bare* log-normal distribution $P(A)$ of the multipliers. Any log-normal distribution is fixed

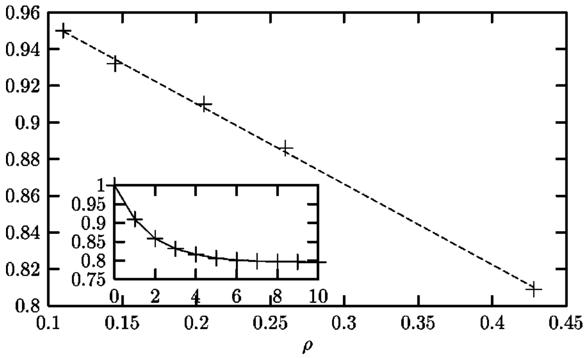


FIG. 3. Results for $\overline{|u_{n+1}|^3|u_n|^2}^{(\sigma)}/\overline{|u_{n+1}|^3}^{(\sigma)}/\overline{|u_n|^3}^{(\sigma)}\overline{|u_n|^5}^{(\sigma)}$ as a function of $\rho_5(\sigma)=\zeta^{(s)}(3)+\zeta^{(s)}(2)-\zeta^{(s)}(5)$. We plot the data measured using the TRMP $(+)$ and the best linear fit $1-0.44\rho_5$. Inset: typical behavior of the fusion rule coefficients $C_n$ vs $n$ obtained from a TRMP stochastic signal.

by two—bare—parameters defining the mean and the variance. We fix the mean of the distribution such as the third-order exponent measured on the TRMP which is consistent with the 4/5 law, $\zeta^{(s)}(3)=1$. Now, we are left with only the variance $\sigma$ of the multipliers probability distribution $P(A)$ as a free parameter. In order to have a control on the ratio,

$$C_2/C_1=\mathcal{K}\frac{\langle|u_{n+1}|^3|u_n|^2\rangle}{\dfrac{\langle|u_{n+1}|^3\rangle}{\langle|u_n|^3\rangle}\langle|u_n|^5\rangle},$$

we may estimate the unknown multiscale correlation functions appearing in the by using the TRMP right-hand side by varying the log-normal distribution:

$$\frac{\langle|u_{n+1}|^3|u_n|^2\rangle}{\dfrac{\langle|u_{n+1}|^3\rangle}{\langle|u_n|^3\rangle}\langle|u_n|^5\rangle}\sim\frac{\overline{|u_{n+1}|^3|u_n|^{2(\sigma)}}}{\dfrac{\overline{|u_{n+1}|^{3(\sigma)}}}{\overline{|u_n|^{3(\sigma)}}}\overline{|u_n|^{5(\sigma)}}},\tag{29}$$

where we have added a superscript $(\sigma)$ in the averages from the TRMP to recall the dependency on the variance of the log-normal distribution. As a result we have a guess on the ratio $C_2/C_1$ at varying $\sigma$ up to the still unknown constant $\mathcal{K}$.

The results of the numerical simulations are shown in Fig. 3 where we plot Eq. (29) as a function of

$$\rho_5(\sigma)=\zeta^{(s)}(3)+\zeta^{(s)}(2)-\zeta^{(s)}(5)$$

in the TRMP.

As one can easily see in Fig. 3, expression (29) is extremely well fitted by a linear behavior:

$$\frac{\overline{|u_{n+1}|^3|u_n|^{2(\sigma)}}}{\dfrac{\overline{|u_{n+1}|^{3(\sigma)}}}{\overline{|u_n|^{3(\sigma)}}}\overline{|u_n|^{5(\sigma)}}}=1-0.44\rho_5(\sigma).\tag{30}$$

This is the third equation, linking the unknowns in Eqs. (23) and (24) and closing the problem. It has been obtained by using the TRMP. This is not yet the end of the story. It is not enough to plug the numerical result (30) into Eqs. (23) and (24) to consistently close the equations. The problem is connected to the possible presence of a *renormalizing* scale-independent stochastic variable in the multiplicative process—the $g_n$ variable already discussed for the passive scalar case. We already discussed that the presence of such a scale-invariant distribution changes only one overall constant in the multiscale behavior. Moreover, we already know that another unknown constant overall $\mathcal{K}$ shows up due to the phases' statistics. Summing the two effects, we can assume that the *true* ratio $C_2/C_1$ which must be plugged into the equation of motion can be estimated by the result of TRMP (30) plus a multiplicative, unknown, constant, $\mathcal{D}$ independent of the intermittency of the model:

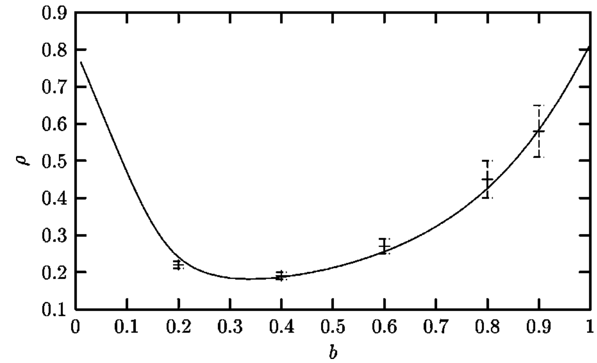


FIG. 4. The values of the fifth-order anomaly $\rho_5=\zeta(3)+\zeta(2)-\zeta(5)$ as a function of $b$ obtained from the TRMP closure approach (continuous line) are compared with the numerical estimate coming from simulations of the shell model $(+)$.

$$\frac{C_2}{C_1}=[1-0.44\rho_5(\sigma)]\mathcal{D}.\tag{31}$$

Using Eqs. (23), (24), and (31), we can compute the fifth-order anomaly $\rho_5=\log_\lambda(R)$ by solving the system of three equations in three unknown, $C_2, C_1, \rho_5$:

$$C_1+b-(1+b)C_1R^{-1}(\rho_5)=0,\tag{32}$$

$$[1-(1+b)xR^{-1}(\rho_5)]C_2+b(1+x)C_1+[R(\rho_5)x-(1+b)]=0,\tag{33}$$

$$C_2/C_1=\mathcal{D}(1-0.44\rho_5).\tag{34}$$

To our knowledge, there are no simple theoretical arguments which can be used in order to fix the value of $\mathcal{D}$. We fix it by assuming that for $b=-0.4$ we recover the value of $\rho_5$ computed in the numerical simulations. It turns out that $\mathcal{D}=0.85$. We can then compute $\rho_5$ for all values of $b$ in the range $-1<b<0$. In Fig. 4 we show the computation of $\rho_5$ obtained by using Eqs. (32)–(34) together with the numerical estimate of $\rho_5$ obtained by simulations of the shell model. As one can see the results are in very good agreement with the numerical data *for the whole range of b*.

In order to validate our results, we have compared the estimate of the anomalous anomaly $\rho_5$ for the values of $b$ and $\lambda$ corresponding to the curve:

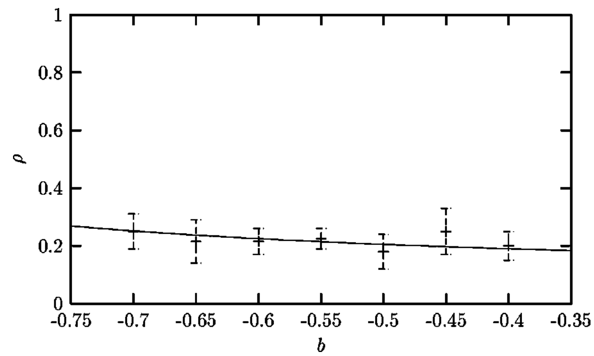


FIG. 5. The values for the fifth-order anomaly $\rho_5=\zeta(3)+\zeta(2)-\zeta(5)$ as a function of $b$ obtained from the TRMP closure approach along the curve $\lambda=1/(1+b)$ (continuous line) are compared with the numerical estimate coming from simulations of the shell model $(+)$.

$$\lambda=\frac{1}{1+b}. \tag{35}$$

Curve (35) is determined by the requirement that the second in-viscid invariant, beside the total energy $\Sigma_n u_n^2$, keeps the physical dimensions of helicity, $\Sigma_n(-)^n k_n|u_n|^2$ [20,21]. It is known [22] that along this curve intermittency stay constant. Numerical estimate of Eq. (30) for different values of $\lambda$ does not show any appreciable difference with respect to what is plotted in Fig. 3. Thus, we can still use Eq. (34) as a numerical estimate of $C_2/C_1$ with *the same value of* $\mathcal{D}$. In Fig. 5 we show the comparison between the results of the closure on the special curve (35) and the values estimated by numerical simulations. Again, the stochastic closure works perfectly, allowing for a precise determination of the anomaly along this curve.

## V. CONCLUSIONS AND DISCUSSIONS

In this paper we have discussed a possible strategy to compute the scaling exponent of a nonlinear shell model. The main idea of this strategy is to assume that the statistical properties of the shell variables $u_n$ can be described in terms of a time-dependent random multiplicative process. A mathematical way to summarize this idea is the following. Let us define the random variable $A_n$ as

$$|u_{n+1}|=A_n|u_n|. \tag{36}$$

Let us also assume that $A_n=\lambda^{h_n}$, where the random variables $h_n$ fluctuate with probability distribution $P_n=Z_n[\exp-V(h_n)]$. If we neglect time dynamics the probability distribution of the shell variables is given by the product $\Pi_k P_k$. Time-dependent random multiplicative process provides well-defined correlations among the random variables for different scales. One can therefore write

$$P(u_1,u_2,\ldots,u_n,\ldots)=Z\cdot\exp[-\Sigma_n V(h_n)+\Sigma_{i,j}R_{ij}h_ih_j]. \tag{37}$$

Once $V(h_n)$ is defined, the time dependency selects a unique value of $R_{i,j}$. Formally we can write $R_{i,j}=\Gamma_{i,j}[V]$. It follows that in order to compute the scaling exponents we need to compute $V(h_n)$. The role of $R_{i,j}$ is crucial because it determines the full shape of the coefficients needed to compute the fusion rules. Using the time average equation of motions for the structure functions, we are therefore able to obtain a functional equation for $V$, whose solutions provide the anomalous scaling exponents of the shell model. In order to understand whether our strategy is providing reasonable results, we have assumed that $V(x)$ is a quadratic function of $x$. Thus, because $\zeta(3)=1$, we have only one unknown to be computed corresponding to the quadratic nonlinearity. With this assumption, the functional equation for $V$ reduces to an equation for one unknown (the quadratic nonlinearity) which we have solved. The results we obtain are in very good agreement with the numerical simulation of the shell model. We want to stress that the same procedure can be applied for the passive scalar advected by a random velocity field (the Kraichnan model) with extremely good results and without any *ad hoc* approximation.

We want to highlight few points which we believe are truely independent of the approximations we did in this paper. (i) The computations of the scaling exponents are feasible if the fusion rules coefficient are known as a function of intermittency [i.e., the function $D(h)$ in the multifractal language]. (ii) The fusion rules coefficient depends on intermittency because of time dynamics. (iii) Time-dependent random multiplicative processes are consistent with the dynamical deterministic structure of the equation of motion and provide a useful tool to compute fusion rules coefficients.

Nevertheless, it is not clear yet, if the stochastic process successfully applied here to close the equation of motion of fourth-order velocity correlation is also the optimal solution for higher-order correlation functions. In other words, one has to face also the possibility that different fluctuations (controlling higher-order correlation functions) are described by different stochastic processes.

In applying our strategy to the shell model we have performed a number of approximations. In particular we consider an important point to generalize our approach in order to properly take into account the phase dynamics in the shell model. Also, the approximation $C_l=D_l$ should be considered as a first order of approximation in order to develop a systematic theory. We also want to highlight two important topics for future research. First of all, we think that it is important to apply our method in the case of the Kraichnan model in two or more space dimensions. In order to perform such a task we need to develop the field theory of time-dependent random multiplicative processes. Also, we need to have a more detailed analytical control of time-dependent random multiplicative processes, following the ideas already discussed in Refs. [8,9].

## ACKNOWLEDGMENT

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